ASYMPTOTIC FORMULA ON AVERAGE PATH LENGTH OF FRACTAL NETWORKS MODELLED ON SIERPINSKI GASKET

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Abstract. In this paper, we introduce a new method to construct evolving networks based on the construction of the Sierpinski gasket. Using self-similarity and renewal theorem, we obtain the asymptotic formula for average path length of our evolving networks.

1. Introduction

The Sierpinski gasket described in 1915 by W. Sierpiński is a classical fractal. Suppose K is the solid regular triangle with vertexes $a_1 = (0, 0)$, $a_2 = (1, 0)$, $a_3 = (1/2, \sqrt{3}/2)$. Let $T_i(x) = x/2+a_i/2$ be the contracting similitude for $i = 1, 2, 3$. Then $T_i : K \to K$ and the Sierpinski gasket E is the self-similar set, which is the unique invariant set [9] of IFS $\{T_1, T_2, T_3\}$, satisfying

$$E = \cup_{i=1}^{3} T_i(E).$$

The Sierpinski gasket is important for the study of fractals, e.g., the Sierpinski gasket is a typical example of post-critically finite self-similar fractals on which the Dirichlet forms and Laplacians can be constructed by Kigami [10, 11], see also Strichartz [14].

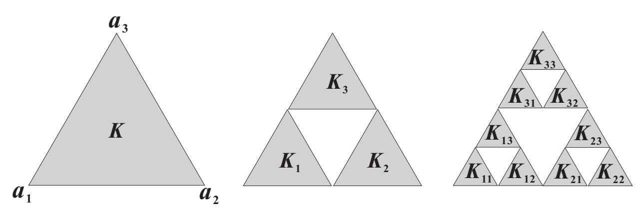

Figure 1. The first two constructions of Sierpinski gasket

For the word $\sigma = i_1 \cdots i_k$ with letters in $\{1, 2, 3\}$, i.e., every letter $i_t \in \{1, 2, 3\}$ for all $t \le k$, we denote by $|\sigma|(= k)$ the length of word σ. Given words $\sigma = i_1 \cdots i_k$ and $\tau = j_1 \cdots j_n$, we call σ a prefix of τ and denote by $\tau \prec \sigma$, if $k < n$ and $i_1 \cdots i_k = j_1 \cdots j_k$. We also write $\tau \preceq \sigma$ if $\tau = \sigma$ or $\tau \prec \sigma$. When $\tau \prec \sigma$ with $|\tau| = |\sigma| - 1$, we say that τ is the father of σ and σ is a child of τ. Given $\sigma = i_1 \cdots i_k$, we write $T_\sigma = T_{i_1} \circ \cdots \circ T_{i_k}$ and $K_\sigma = T_\sigma(K)$ which is a solid regular triangle with side length $2^{-|\sigma|}$. For notational convenience, we write $K_\emptyset = K$ with empty word $\emptyset$. We also denote $|\emptyset| = 0$. If $\tau \prec \sigma$, then $K_\sigma \subset K_\tau$. For solid triangle K_σ with word

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σ, we denote by ∂K_σ its boundary consisting of 3 sides, where every side is a line segment with side length $2^{-|\sigma|}$.

Complex networks arise from natural and social phenomena, such as the Internet, the collaborations in research, and the social relationships. These networks have in common two structural characteristics: the small-world effect and the scale-freeness (*power-law* degree distribution), as indicated, respectively, in the seminal papers by Watts and Strogatz [16] and by Barabási and Albert [2]. In fact complex networks also exhibit *self-similarity* as demonstrated by Song, Havlin and Makse [13] and fractals possess the feature of *power law* in terms of their fractal dimension (e.g. see [6]). Recently self-similar fractals are used to model evolving networks, for example, in a series of papers, Zhang et al. [18, 19, 8] use the Sierpinski gasket to construct evolving networks. There are also some complex networks modelled on self-similar fractals, for example, Liu and Kong [12] and Chen et al. [4] study Koch networks, Zhang et al. [17] investigate the networks constructed from Vicsek fractals. See also Dai and Liu [5], Sun et al. [15] and Zhou et al. [20].

In the paper, we introduce a new method to construct evolving networks modelled on Sierpinski gasket and study the asymptotic formula for average path length.

Since E is connected, we can construct the network from geometry as follows.

Fix an integer t, we consider a network G_t with vertex set $V_t = \{\sigma : 0 \leq |\sigma| \leq t\}$ where $\#V_t = 1 + 3 + ... + 3^t = \frac{1}{2}(3^{t+1} - 1)$. For the edge set of G_t, there is a unique edge between distinct words σ and τ (denoted by $\tau \sim \sigma$) if and only if

$$\partial K_\sigma \cap \partial K_\tau \neq \varnothing. \tag{1.1}$$

We can illustrate the geodesic paths in Figure 2 for $t = 3$. We have $233 \sim 32 \sim 312$ since $\partial K_{233} \cap \partial K_{32} = \{C\}$ and $\partial K_{32} \cap \partial K_{312} = \{F\}$. We also get another geodesic path from 233 to 312 : $233 \sim 3 \sim 312$ since $\partial K_{233} \cap \partial K_3 = \{C\}$ and $\partial K_3 \cap \partial K_{312} = [G, F]$, the line segment between G and F. We also have some geodesic paths from 21 to 312 : $21 \sim 1 \sim 311 \sim 312$, $21 \sim \emptyset \sim 3 \sim 312$ and $21 \sim 2 \sim 32 \sim 312$. Also we have $132 \sim 1 \sim \emptyset$ but $132 \not\sim \emptyset$, then the geodesic distance between 132 and $\emptyset$ is 2.

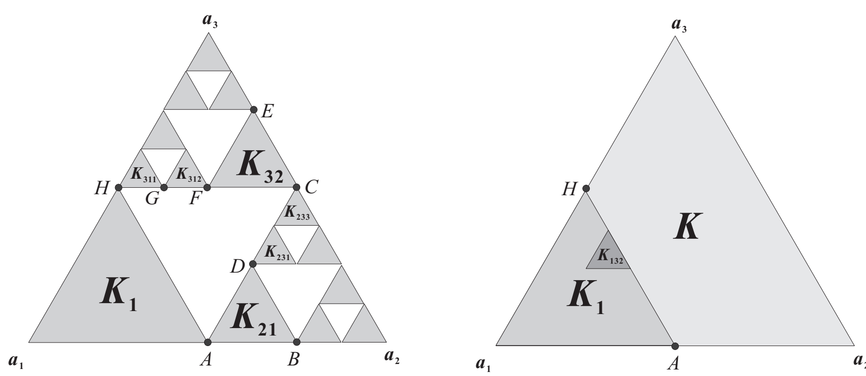

FIGURE 2.

In Figure 2, $312 \not\sim \emptyset$, $231 \not\sim \emptyset$ and $132 \not\sim \emptyset$. In fact, by observation we have

Claim 1. *Suppose $\sigma \prec \tau$ with $\tau = \sigma\beta$. Then $\sigma \sim \tau$ if and only if there are at most two letters in β. In particular, if $\sigma \prec \tau$ and $|\tau| - |\sigma| \leq 2$, then $\sigma \sim \tau$.*

For example, 123123 and 1231 are neighbors, but 123123 and 123 are not.

For every t, we denote $d_t(\sigma,\tau)$ the geodesic distance on V_t. Let

$$\bar{D}(t)=\frac{\sum_{\sigma\neq\tau\in V_t} d_t(\sigma,\tau)}{\#V_t(\#V_t-1)/2}$$

be the average path length of the complex network V_t.

We can state our main result as follows.

Theorem 1. *We have the asymptotic formula*

$$\lim_{t\to\infty}\frac{\bar{D}(t)}{t}=\frac{4}{9}. \tag{1.2}$$

Remark 1. *Since $t\propto \ln(\#V_t)$, Theorem 1 implies that the evolving networks G_t have small average path length, namely $\bar{D}(t)\propto\ln(\#V_t)$.*

The paper is organized as follows. In Section 2 we give notations and sketch of proof for Theorem 1,consisting of four steps. In sections 3-6, we will provide details for the four steps respectively. Our main techniques come from the self-similarity and the renewal theorem.

2. Sketch of proof for Theorem 1

We will illustrate our following four steps needed to prove Theorem 1.

Step 1. We calculate the geodesic distance between a word and the empty word.

Given a small solid triangle Δ, we can find a maximal solid triangle Δ' which contains Δ and their boundaries are touching. Translating into the language of words, for a given word $\sigma\neq\emptyset$, we can find a unique shortest word $f(\sigma)$ such that $f(\sigma)\prec\sigma$ and $f(\sigma)\sim\sigma$. For a word $\sigma=\tau_2\tau_1$, where τ_1 is the maximal suffix with at most two letters appearing, using Claim 1 we have $f(\sigma)=\tau_2$. Iterating f again and again, we obtain a sequence $\sigma\sim f(\sigma)\sim\cdots\sim f^{n-1}(\sigma)\sim f^n(\sigma)=\emptyset$. Let

$$\omega(\sigma)=\min\{n: f^n(\sigma)=\emptyset\}.$$

In particular, we define $\omega(\emptyset)=0$. For $\sigma=112113112312=(112)(11311)(23)(12)$, we have $f(\sigma)=(112)(11311)(23)$, $f^2(\sigma)=(112)(11311)$, $f^3(\sigma)=(112)$ and $f^4(\sigma)=\emptyset$. Then $\omega(\sigma)=4$. We will prove in Section 3

Proposition 1. *For $\sigma\in V_t$, we have $d_t(\sigma,\emptyset)=\omega(\sigma)$.*

In fact, this proposition shows that $d_t(\sigma,\emptyset)$ is independent of the choice of t whenever $t\geq|\sigma|$. In this case, we also write $d(\sigma,\emptyset)$. Write

$$L(\tau)=d(\tau,\emptyset)-1 \text{ for } \tau\neq\emptyset \text{ and } L(\emptyset)=0.$$

Then $L(\tau)$ is independent of t, in fact, $L(\tau)$ is the minimal number of moves for K_τ to touch the boundary of K.

Step 2. Given $m\geq 1$, we consider the average geodesic distance between the empty word and word of length m and set

$$\bar{\alpha}_m=\frac{\sum_{|\sigma|=m} d(\sigma,\emptyset)}{\#\{\sigma:|\sigma|=m\}}-1=\frac{\sum_{|\sigma|=m} L(\sigma)}{\#\{\sigma:|\sigma|=m\}},$$

and $\bar{\alpha}_0=0$, we will obtain the limit property of $\bar{\alpha}_m/m$ as $m\to\infty$ in Section 4.

In fact, by the **Jordan curve theorem**, we can obtain

$$L(\tau)+L(\sigma)\leq L(\tau\sigma)\leq L(\tau)+L(\sigma)+1. \tag{2.1}$$

From (2.1), we can verify $\{\bar{\alpha}_m\}_m$ is superadditive which implies

Proposition 2. $\lim_{m\to\infty} \bar{\alpha}_m/m = \sup(\bar{\alpha}_m/m) < \infty$.

Denote

$$\alpha^* = \sup(\bar{\alpha}_m/m) = \lim_{m\to\infty} \bar{\alpha}_m/m.$$

Step 3. We obtain the asymptotic formula of $\bar{D}(t)$ in terms of α^*.

Using the similarity of Sierpinski gasket, e.g., for $i = 1, 2, 3$,

$$d(i\sigma, i\tau) = d(\sigma, \tau),$$

and for $i \neq j$,

$$L(\sigma) + L(\tau) \leq d(i\sigma, j\tau) \leq (L(\sigma) + 1) + (L(\tau) + 1) + 1,$$

we will prove the following in Section 5

Proposition 3. $\lim_{t\to\infty} \frac{\bar{D}(t)}{t} = 2\alpha^*$.

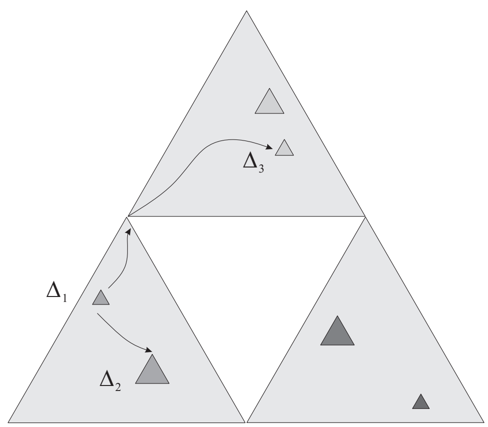

FIGURE 3. The typical geodesic path between Δ_1 and Δ_3

As illustrated in Figure 3, Proposition 3 shows that the *typical* geodesic path is the geodesic path between Δ_1 and Δ_3 whose first letters of codings are different. On the other hand, for example the geodesic path between Δ_1 and Δ_2 with the same first letter will give negligible contribution to $\bar{D}(t)$. Using $L(\sigma) + L(\tau) \leq d(i\sigma, j\tau) \leq L(\sigma) + L(\tau) + 3$ with $i \neq j$, we obtain that $d(i\sigma, j\tau) \approx L(\sigma) + L(\tau)$, ignoring the terms like $d(i\sigma, i\tau)$, we have

$$\frac{\bar{D}(t)}{t} \approx 2 \cdot \frac{1}{t} \cdot \frac{\sum_{|\tau|\leq t-1} L(\tau)}{\#\{\tau : |\tau| \leq t-1\}},$$

where $\frac{\sum_{|\tau|\leq t-1} L(\tau)}{\#\{\tau:|\tau|\leq t-1\}}$ is the average value of $L(\tau)$. Using Stolz theorem, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{2\sum_{|\tau|\leq t-1} L(\tau)}{t \cdot \#\{\tau : |\tau| \leq t-1\}} &= \lim_{t\to\infty} \frac{2\sum_{|\tau|=t-1} L(\tau)}{3^{t-1}t + \frac{3^t}{6} - \frac{1}{2}} \\ &= \lim_{t\to\infty} \frac{2}{t} \cdot \frac{\sum_{|\tau|=t-1} L(\tau)}{\#\{\tau : |\tau| = t-1\}} \qquad (2.2) \\ &= \lim_{t\to\infty} \frac{2\bar{\alpha}_{t-1}}{t} = 2\alpha^*. \end{aligned}$$

Step 4. Using the renewal theorem, we will prove in Section 6

Proposition 4. $\alpha^* = 2/9$.

By programming, we have

$t =$	300	400	500	600	700	800
$\bar{\alpha}_t/t =$	$0.2207\cdots$	$0.2211\cdots$	$0.2213\cdots$	$0.2214\cdots$	$0.2215\cdots$	$0.2216\cdots$

which is in line with $\alpha^* = 2/9 = 0.2222\cdots$.

In fact, suppose $\Sigma = \{\cdots x_2 x_1 : x_i = 1, 2 \text{ or } 3 \text{ for all } t\}$ is composed of infinite words with letters in $\{1, 2, 3\}$. Then we have a natural mass distribution μ on Σ such that for any word σ of length k,

$$\mu(\{\cdots x_k \cdots x_1 : x_k \cdots x_1 = \sigma\}) = 1/3^k.$$

For any word σ, let $\#(\sigma)$ denote the cardinality of letters appearing in word σ. For μ-almost all x, let

$$S(\cdots x_p x_{p-1} \cdots x_1) = p \text{ if } \#(x_{p-1} \cdots x_1 1) = 2 \text{ and } \#(x_p x_{p-1} \cdots x_1 1) = 3.$$

Then $\mathbb{E}(S) = \sum_{k=2}^{\infty} k \cdot \mu\{x : S(x) = k\} = \sum_{k=2}^{\infty} k \cdot (2^k - 2)/3^k = 9/2 < \infty$.

For μ-almost all $x = \cdots x_2 x_1$, suppose $x_0 = 1$ and $p_0 = 0$ and there is an infinite sequence $\{p_n\}_{n \geq 0}$ of integers such that $p_{n+1} > p_n$ for all n and

$$x1 = \cdots x_{p_3} \cdots x_{p_2} \cdots x_{p_1} \cdots x_1 x_0$$

satisfying $\#(x_{p_n} \cdots x_{p_{n-1}}) = 3$ and $\#(x_{(p_n-1)} \cdots x_{p_{n-1}}) = 2$ for all n. We then let

$$S_i(x) = p_i - p_{i-1} \text{ for any } i \geq 1.$$

Since x_{p_n} is uniquely determined by word $x_{(p_n-1)} \cdots x_{p_{n-1}}$, and μ is symmetric for letters in $\{1, 2, 3\}$, we find that $\{S_i\}_i$ is a sequence of positive independent identically distributed random variables with $S_1 = S$. For example, for $x = \cdots 321223121$, then $x1 = \cdots 3(21)(223)(1211)$ and $S_1(x) = S(x) = 4$, $S_2(x) = 3$, $S_3(x) = 2, \cdots$.

Let $J_n = S_1 + \cdots + S_n$ and $Y_t = \sup\{n : J_n \leq t\}$. Then $J_n = p_n$ and $Y_t = \max\{n : p_n \leq t\}$. By the elementary renewal theorem, we have

$$\frac{\mathbb{E}(Y_t)}{t} \to \frac{1}{\mathbb{E}(S)} = \frac{2}{9} = 0.222\cdots.$$

Using the following estimates (Lemma 8 in Section 6)

$$\begin{aligned}
\liminf_{t\to\infty} \frac{\sum_{|\sigma|=t} d(\sigma, \emptyset)}{t3^t} &\geq \liminf_{t\to\infty} \frac{\sum_{k=2}^{t-1} \mathbb{E}(Y_{t-k}) \frac{2^k-2}{3^k}}{t}, \\
\limsup_{t\to\infty} \frac{\sum_{|\sigma|=t} d(\sigma, \emptyset)}{t3^t} &\leq \limsup_{t\to\infty} \frac{\sum_{k=2}^{t-1} \mathbb{E}(Y_{t-k}) \frac{2^k-2}{3^k}}{t},
\end{aligned}$$

we can prove that $\alpha^* = \lim_{t\to\infty} \frac{\sum_{|\sigma|=t} d(\sigma,\emptyset)}{t3^t} = \lim_{t\to\infty} \frac{\mathbb{E}(Y_t)}{t} = 2/9$.

Theorem 1 follows from Propositions 3 and 4.

3. Basic formulas on geodesic distance

3.1. Criteria of neighbor.

Given distinct words σ and τ with $|\sigma|, |\tau| \le t$, we give the following **criteria** to test whether they are neighbors or not. At first, we delete the common prefix of σ and τ, say $\sigma = \beta\sigma'$ and $\tau = \beta\tau'$ where the first letters of σ' and τ' are different. We can distinguish two cases:

Case 1. If one of σ' and τ' is the empty word, say $\sigma' = \emptyset$, then σ and τ are not neighbors if and only if every letter $i \in \{1, 2, 3\}$ appears in the word τ'.

Case 2. If neither σ' nor τ' is the empty word, say i, j the first letters of σ' and τ' respectively, then σ and τ are neighbors if and only if

$$\sigma' = i[j]^k \text{ and } \tau' = j[i]^{k'} \text{ with } k, k' \ge 0.$$

We say that σ and τ are neighbors of type 1 or 2 according to Case 1 or 2 respectively.

3.2. Estimates on distance.

The first lemma show the *self-similarity* of geodesic distance.

Lemma 1. *If $\sigma = i\sigma'$ and $\tau = i\tau'$ with $i \in \{1, 2, 3\}$, then*

$$d_t(\sigma, \tau) = d_t(i\sigma', i\tau') = d_{t-1}(\sigma', \tau').$$

Consequently, given any word $i_1 \cdots i_k$,

$$d_t(i_1 \cdots i_k\sigma', i_1 \cdots i_k\tau') = d_{t-k}(\sigma', \tau') \text{ for all } \sigma', \tau'.$$

Proof. Fix the letter i, we define $g : V_t \to V_t$ by

$$g(\beta) = \begin{cases} \beta & \text{if } i \preceq \beta, \\ i & \text{otherwise.} \end{cases}$$

If we give a shortest sequence $(\sigma =)\sigma^1 \sim \sigma^2 \sim \cdots \sim \sigma^k (= \tau)$ in G_t, then $\sigma = g(\sigma^1) \simeq g(\sigma^2) \simeq \cdots \simeq g(\sigma^k) = \tau$ is also a sequence and all word $\{g(\sigma^i)\}_{i=1}^k$ have the same first letter i. Deleting the first letter i, we get a $(\simeq)$-sequence from σ' to τ' in G_{t-1}. Hence $d_t(\sigma, \tau) \ge d_{t-1}(\sigma', \tau')$.

On the other hand, for given $(\sim)$-sequence from σ' to τ' in G_{t-1}, by adding the first letter i, we obtain a $(\sim)$-sequence σ to τ in G_t which implies $d_t(\sigma, \tau) \le d_{t-1}(\sigma', \tau')$. The lemma follows. □

Given t, let $L_t(\sigma) = d_t(\sigma, \emptyset) - 1$ for $\sigma \ne \emptyset$. The second lemma shows that $L_t(\sigma)$ is independent of t whenever $t \ge |\sigma|$. We can write $L(\sigma)$.

Lemma 2. *For $\sigma, \tau \in V_k$, $d_t(\sigma, \tau) = d_k(\sigma, \tau)$. As a result, $L_t(\sigma) = L_{|\sigma|}(\sigma)$.*

Proof. Fix $k \le t$. Let $h : V_t \to V_k$ be defined by

$$h(\beta) = \begin{cases} \beta & \text{if } |\beta| \le k, \\ i_1 \cdots i_k & \text{if } |\beta| > k \text{ and } \beta = i_1 \cdots i_k \cdots i_{|\beta|}. \end{cases}$$

Given $\sigma, \tau \in V_k$, if we give a shortest sequence $(\sigma =)\sigma^1 \sim \sigma^2 \sim \cdots \sim \sigma^k (= \tau)$ in G_t, by criteria of neighbor we obtain that $\sigma = h(\sigma^1) \simeq h(\sigma^2) \simeq \cdots \simeq h(\sigma^k) = \tau$ is also a $(\simeq)$-sequence and all words in V_k. Therefore, $d_t(\sigma, \tau) \ge d_k(\sigma, \tau)$. On the other hand, any $(\sim)$-sequence from σ to τ in G_k is also a $(\sim)$-sequence in G_k, that means $d_t(\sigma, \tau) \le d_k(\sigma, \tau)$. The lemma follows. □

By the *Jordan curve theorem*, when a point in the *interior* moves to the *exterior*, it must touch the *boundary*. Therefore, we have the following

Lemma 3. *For any word $\tau\sigma$ with $\tau, \sigma \neq \emptyset$, we have*

$$L(\tau) + L(\sigma) \leq L(\tau\sigma) \leq L(\tau) + L(\sigma) + 1.$$

Proof. In fact, using triangle inequality and Lemmas 1 and 2, we have

$$\begin{aligned} L(\tau\sigma) = d_t(\tau\sigma, \emptyset) - 1 &\leq d_t(\tau\sigma, \tau) + d_t(\tau, \emptyset) - 1 \\ &= d_{t-|\tau|}(\sigma, \emptyset) + d_t(\tau, \emptyset) - 1 \\ &= L(\sigma) + L(\tau) + 1. \end{aligned}$$

On the other hand, using the self-similarity in Lemma 1, the minimal number of moves for $K_{\tau\sigma}$ to touch the boundary of K_τ is $L(\sigma)$, and $L(\tau)$ is the minimal number of moves for K_τ to touch the boundary of K, by the Jordan curve theorem, there are at least $L(\sigma) + L(\tau)$ moves for $K_{\tau\sigma}$ to touch the boundary of K, that means $L(\tau) + L(\sigma) \leq L(\tau\sigma)$. □

3.3. **Proof of Proposition 1.**

Suppose ω and f are defined as in Section 2. By the definition of f, if $\tau' \preceq \sigma'$, then we have $f(\tau') \preceq f(\sigma')$. Therefore we have

Claim 2. *If $f(\sigma) \preceq f(\tau) \preceq \sigma$, then $f^k(\sigma) \preceq f^k(\tau) \preceq f^{k-1}(\sigma)$ for all $k \geq 0$. As a result, $|\omega(\sigma) - \omega(\tau)| \leq 1$.*

Example 1. *Let $i = 1$, $j = 3$ and $\sigma = \beta 321211$ for some word β, we have $f(\sigma i) = \beta 3$ and $f(\sigma j) = \beta 3212$. Then*

$$f(\sigma i) \preceq f(\sigma j) \preceq \sigma i,$$

and thus $|\omega(\sigma i) - \omega(\sigma j)| \leq 1$. For $\beta = 132$ and $\beta' = \beta 1211$ with $\beta \sim \beta'$, we have $f(\beta) \preceq f(\beta') \preceq \beta$, then $|\omega(\beta) - \omega(\beta')| \leq 1$.

Lemma 4. *If $\sigma \sim \tau$, then $\omega(\sigma) \geq \omega(\tau) - 1$.*

Proof. By the criteria of neighbor, without loss of generality, we only need to deal with three cases: (1) $\sigma \prec \tau$; (2) $\sigma i[j]^p$ and $\sigma j[i]^q$ with $i \neq j$ and $p, q \geq 1$; (3) σi and $\sigma j[i]^q$ with $i \neq j$ and $q \geq 0$. For case (2), it is clear that

$$\omega(\sigma i[j]^p) = \omega(\sigma j[i]^q) = \omega(\sigma ij).$$

For cases (1) and (3), we only use Claim 2 as in Example 1 above. □

Proof of Proposition 1. As shown in Section 2, we can find the following path from σ to the empty word $\emptyset$:

$$\sigma \sim f(\sigma) \sim f^2(\sigma) \sim \cdots \sim f^{\omega(\sigma)}(\sigma_k) = \emptyset.$$

That means $d_t(\sigma, \emptyset) \leq \omega(\sigma)$.

It suffices to show $d_t(\sigma, \emptyset) \geq \omega(\sigma)$. Suppose on the contrary, if we give a sequence

$$\sigma = \sigma_0 \sim \sigma_1 \sim \sigma_2 \sim \cdots \sim \sigma_k = \emptyset \text{ with } k \leq \omega(\sigma) - 1,$$

then $\omega(\sigma_{i+1}) \geq \omega(\sigma_i) - 1$ for all i by Lemma 4. Therefore, $0 = \omega(\emptyset) \geq \omega(\sigma) - k > 0$ which is impossible. That means $d_t(\sigma, \emptyset) = \omega(\sigma)$. □

4. Average geodesic distance to empty word

We first recall some notations. For every t, we denote $d_k(\sigma,\tau)$ the geodesic distance on V_k. Given $k\geq 0$, let $L_k(\emptyset)=0$ and

$$L_k(\sigma)=d_k(\sigma,\emptyset)-1 \text{ for } \sigma\in V_k \text{ with } \sigma\neq\emptyset.$$

As shown in (2.2), for average geodesic distance $\frac{\sum_{|\tau|\leq t-1}L(\tau)}{\#\{\tau:|\tau|\leq t-1\}}+1$ to the empty word, when we estimate $\frac{1}{t}\frac{\sum_{|\tau|\leq t-1}L(\tau)}{\#\{\tau:|\tau|\leq t-1\}}$, it is important for us to estimate $\bar{\alpha}_k/k$, where

$$\bar{\alpha}_k=\frac{\sum_{|\sigma|=k}L_k(\sigma)}{\#\{\sigma:|\sigma|=k\}} \text{ for } k\geq 0.$$

Lemma 5. *For any $k_1,k_2\geq 1$, we have*

$$\bar{\alpha}_{k_1}+\bar{\alpha}_{k_2}\leq\bar{\alpha}_{k_1+k_2}\leq\bar{\alpha}_{k_1}+\bar{\alpha}_{k_2}+1. \tag{4.1}$$

In particular, $\{\bar{\alpha}_k\}_k$ is non-decreasing, i.e.,

$$\bar{\alpha}_{k+1}\geq\bar{\alpha}_k \text{ for all } k. \tag{4.2}$$

As a result, for any positive integers q and k we obtain that

$$\bar{\alpha}_k\geq\bar{\alpha}_q\Big[\frac{k}{q}\Big]\geq\frac{\bar{\alpha}_q}{q}(k-q+1), \tag{4.3}$$

Proof. We obtain that

$$\bar{\alpha}_{k_1+k_2}=\frac{\sum_{|\tau|=k_1}\sum_{|\sigma|=k_2}L_{k_1+k_2}(\tau\sigma)}{\#\{\tau:|\tau|=k_1\}\cdot\#\{\sigma:|\sigma|=k_2\}}.$$

If $|\tau|=k_1$ and $|\sigma|=k_2$, using Lemma 3, we have

$$L_{k_1}(\tau)+L_{k_2}(\sigma)\leq L_{k_1+k_2}(\tau\sigma)\leq L_{k_1}(\tau)+L_{k_2}(\sigma)+1, \tag{4.4}$$

which implies

$$\begin{aligned}\bar{\alpha}_{k_1+k_2} &\geq \frac{\sum_{|\tau|=k_1}\sum_{|\sigma|=k_2}(L_{k_1}(\tau)+L_{k_2}(\sigma))}{\#\{\tau:|\tau|=k_1\}\cdot\#\{\sigma:|\sigma|=k_2\}}=\bar{\alpha}_{k_1}+\bar{\alpha}_{k_2},\\ \bar{\alpha}_{k_1+k_2} &\leq \frac{\sum_{|\tau|=k_1}\sum_{|\sigma|=k_2}(L_{k_1}(\tau)+L_{k_2}(\sigma)+1)}{\#\{\tau:|\tau|=k_1\}\cdot\#\{\sigma:|\sigma|=k_2\}}=\bar{\alpha}_{k_1}+\bar{\alpha}_{k_2}+1,\end{aligned}$$

then (4.1) follows. In particular, we have $\bar{\alpha}_{k+1}\geq\bar{\alpha}_k+\bar{\alpha}_1\geq\bar{\alpha}_k$.

Using (4.1) repeatedly, we have $\bar{\alpha}_{qm}\geq\bar{\alpha}_{q(m-1)}+\bar{\alpha}_q\geq\cdots\geq m\bar{\alpha}_q$. It follows from (4.2) that

$$\bar{\alpha}_{qm+(q-1)}\geq\bar{\alpha}_{qm+(q-2)}\geq\cdots\geq\bar{\alpha}_{qm+1}\geq\bar{\alpha}_{qm}\geq m\bar{\alpha}_q$$

which implies $\bar{\alpha}_k\geq\bar{\alpha}_q[\frac{k}{q}]\geq\frac{\bar{\alpha}_q}{q}(k-q+1)$. □

Proof of Proposition 2. Since $\{\bar{\alpha}_m\}_m$ is superadditive, i.e., $\bar{\alpha}_{k_1+k_2}\geq\bar{\alpha}_{k_1}+\bar{\alpha}_{k_2}$, by Fekete's superadditive lemma ([7]), the limit $\lim\limits_{m\to\infty}\frac{\bar{\alpha}_m}{m}$ exists and is equal to $\sup_m\frac{\bar{\alpha}_m}{m}$. We shall verify that $\lim\limits_{m\to\infty}\frac{\bar{\alpha}_m}{m}<+\infty$.

Fix an integer q. For any $p=0,1,\cdots,(q-1)$, using (4.1) and (4.2), we have $\bar{\alpha}_{qk+p}\leq\bar{\alpha}_{q(k+1)}\leq(k+1)\bar{\alpha}_q+k$. Letting $k\to\infty$, we have $\limsup\limits_{m\to\infty}\frac{\bar{\alpha}_m}{m}\leq\frac{\bar{\alpha}_q}{q}+\frac{1}{q}$. □

Set $\alpha^*=\lim_{m\to\infty}\frac{\bar{\alpha}_m}{m}=\sup_m\frac{\bar{\alpha}_m}{m}$.

5. Asymptotic formula

Now we will investigate

$$\kappa_t = \frac{\sum_{\sigma\in V_t} L(\sigma)}{\#V_t} = \frac{\sum_{k=0}^t \sum_{|\sigma|=k} L_k(\sigma)}{\sum_{k=0}^t \#\{\sigma : |\sigma| = k\}} = \frac{\sum_{k=0}^t \bar{\alpha}_k 3^k}{\sum_{k=0}^t 3^k},$$

where we let $L(\emptyset) = \bar{\alpha}_0 = \frac{\bar{\alpha}_0}{0} = 0$. At first, we have

$$\kappa_t \leq \left(\sup_{m\geq 1} \frac{\bar{\alpha}_m}{m}\right) \frac{\sum_{k=0}^t k3^k}{\sum_{k=0}^t 3^k} = \alpha^* \frac{\sum_{k=0}^t k3^k}{\sum_{k=0}^t 3^k} \leq (\alpha^*\chi(t))t, \tag{5.1}$$

where

$$\chi(t) = \frac{\sum_{k=0}^t k3^k}{t\sum_{k=0}^t 3^k} = \frac{(3t-\frac{3}{2}) + \frac{3}{2}\frac{1}{3^t}}{(3t)(1-\frac{1}{3^{t+1}})} \leq 1, \tag{5.2}$$

since $(3t-\frac{3}{2}) + \frac{3}{2}\frac{1}{3^t} - (3t)(1-\frac{1}{3^{t+1}}) = \frac{1}{2}\left(2t - 3^{t+1} + 3\right)\frac{1}{3^t} < 0$ for any $t \geq 1$. We also have

$$\lim_{t\to\infty} \chi(t) = 1. \tag{5.3}$$

On the other hand, using $\bar{\alpha}_k \geq \frac{\bar{\alpha}_q}{q}(k-q+1)$ in (4.3), for any t we have

$$\kappa_t \geq \frac{\sum_{k=0}^t \frac{\bar{\alpha}_q(k-q+1)}{q} 3^k}{\sum_{k=0}^t 3^k} \geq \frac{\bar{\alpha}_q}{q}\left(\frac{\sum_{k=0}^t k\cdot 3^k}{\sum_{k=0}^t 3^k} - q + 1\right),$$

that is

$$\kappa_t \geq \frac{\bar{\alpha}_q}{q}(\chi(t)\cdot t - q + 1). \tag{5.4}$$

Denote

$$\pi_t = \sum_{\sigma,\tau\in V_t} d_t(\sigma,\tau), \qquad \mu_t = \sum_i \sum_{i\preceq\sigma, i\preceq\tau} d_t(\sigma,\tau),$$
$$\lambda_t = \sum_{\sigma\in V_t} d_t(\sigma,\emptyset), \qquad \nu_t = \sum_{i\neq j} \sum_{i\preceq\sigma, j\preceq\tau} d_t(\sigma,\tau).$$

Then

$$\pi_t = \mu_t + \lambda_t + \nu_t.$$

By Lemma 1, we have

$$\mu_t = 3\sum_{\sigma',\tau'\in V_{t-1}} d_{t-1}(\sigma',\tau') = 3\pi_{t-1},$$

and thus

$$\pi_t = 3\pi_{t-1} + \lambda_t + \nu_t. \tag{5.5}$$

(1) The estimate of λ_t : Using (5.1)-(5.2), we have

$$\begin{aligned} \lambda_t &= \left(\frac{\sum_{\sigma\in V_t} d_t(\sigma,\emptyset)}{\#V_t}\right)\#V_t \\ &= \left(\frac{\sum_{\sigma\in V_t} L_t(\sigma,\emptyset)}{\#V_t} + \frac{\sum_{\sigma\neq\emptyset} 1}{\#V_t}\right)\#V_t \\ &= \kappa_t\#V_t + (\#V_t - 1) \leq \alpha^* t(\#V_t) + (\#V_t - 1). \end{aligned} \tag{5.6}$$

(2) The estimate of ν_t : For $\sigma = i\sigma'$, $\tau = j\tau'$ with $i \neq j$, by Lemma 1 we have

$$d_t(\sigma,\tau) \leq d_t(i\sigma', i) + d_t(j\tau', j) + d_t(i,j) = d_{t-1}(\sigma',\emptyset) + d_{t-1}(\tau',\emptyset) + 1. \tag{5.7}$$

Notice that

$$d_{t-1}(\sigma',\emptyset) \le L(\sigma')+1 \text{ for all } \sigma' \in V_{t-1}, \tag{5.8}$$

since $d_{t-1}(\sigma',\emptyset) = L(\sigma')+1$ for $\sigma' \neq \emptyset$ and (5.8) is also true for $\sigma' = \emptyset$.

Using (5.1)-(5.2) and (5.7)-(5.8), we have the following upper bound of ν_t.

$$\begin{aligned} \nu_t &= \sum\nolimits_{i\neq j}\sum\nolimits_{i\preceq\sigma, j\preceq\tau} d_t(\sigma,\tau) \\ &\le C_3^2 \sum\nolimits_{\sigma',\tau'\in V_{t-1}} (d_{t-1}(\sigma',\emptyset)+d_{t-1}(\tau',\emptyset)+1) \\ &\le 3(\#V_{t-1})^2 + 6(\#V_{t-1})^2 \frac{\sum_{\sigma'\in V_{t-1}}(L(\sigma')+1)}{\#V_{t-1}} \\ &\le 3(\#V_{t-1})^2 + 6(\#V_{t-1})^2(\kappa_{t-1}+1) \\ &\le 9(\#V_{t-1})^2 + 6(\#V_{t-1})^2 \cdot \alpha^* \cdot (t-1). \end{aligned} \tag{5.9}$$

On the other hand, for $i \neq j$, using the Jordan curve theorem, we have

$$d_t(i\sigma', j\tau') \ge L_{t-1}(\sigma') + L_{t-1}(\tau').$$

Then we obtain the following lower bound of ν_t.

$$\begin{aligned} \nu_t &= \sum\nolimits_{i\neq j}\sum\nolimits_{i\preceq\sigma, j\preceq\tau} d_t(\sigma,\tau) \\ &\ge C_3^2 \sum\nolimits_{\sigma',\tau'\in V_{t-1}} (L_{t-1}(\sigma')+L_{t-1}(\tau')) \\ &\ge 6\frac{\sum_{\sigma'\in V_{t-1}} L_{t-1}(\sigma')}{(\#V_{t-1})}(\#V_{t-1})^2 \\ &\ge 6\kappa_{t-1}(\#V_{t-1})^2 \\ &\ge 6\left(\frac{\kappa_{t-1}}{t-1}\right)(\#V_{t-1})^2(t-1), \end{aligned} \tag{5.10}$$

where

$$\frac{\kappa_{t-1}}{t-1} \to \alpha^* \text{ as } t \to \infty \tag{5.11}$$

since

$$\frac{\kappa_{t-1}}{t-1} = \frac{\sum_{k=0}^{t-1}\left(\frac{\bar{\alpha}_k}{k}\right)k3^k}{(t-1)\sum_{k=0}^{t-1}3^k}$$

with $\lim\limits_{k\to\infty}\frac{\bar{\alpha}_k}{k} = \alpha^*$ and $\lim\limits_{t\to\infty}\frac{\sum_{k=0}^{t-1}k3^k}{(t-1)\sum_{k=0}^{t-1}3^k} = \lim\limits_{t\to\infty}\chi(t-1) = 1.$

Proof of Proposition 3.

(i) Upper bound of π_t : Using (5.6) and (5.9), we have

$$\lambda_t + \nu_t \le \psi(t) + 6(\#V_{t-1})^2\alpha^*(t-1), \tag{5.12}$$

where $\psi(t) = \alpha^* t(\#V_t) + (\#V_t - 1) + 9(\#V_{t-1})^2$.

Fix an integer q. Using (5.5) and (5.12) again and again, for $t > q$ we have

$$\begin{aligned} \pi_t &\le 3\pi_{t-1} + \psi(t) + 6\alpha^*(t-1)(\#V_{t-1})^2 \\ &\le 3^2\pi_{t-2} + (\psi(t)+3\psi(t-1)) + 6\alpha^*\left((t-1)(\#V_{t-1})^2 + 18\alpha^*(t-2)(\#V_{t-2})^2\right) \\ &\le \cdots \le 3^{t-q}\pi_q + \sum_{k=q}^{t-1}3^{t-k-1}\psi(k+1) + 6\alpha^*\sum_{k=q}^{t-1}3^{t-k-1}k(\#V_k)^2. \end{aligned}$$

We can check that

$$\frac{3^{t-q}\pi_q + \sum_{k=q}^{t-1} 3^{t-k-1}\psi(k+1)}{t(\#V_t)^2} \to 0 \text{ as } t \to \infty.$$

In fact, we only need to estimate

(i) $\frac{\sum_{k=q}^{t-1} 3^{t-k-1}(k+1)(\#V_{k+1})}{t(\#V_t)^2} = \frac{\sum_{k=q+1}^{t} 3^{t-k} k \frac{3^{k+1}-1}{2}}{t(\frac{3^{t+1}-1}{2})^2} \le \frac{3}{2} \frac{\frac{t(t+1)}{2} 3^t}{t(\frac{3^{t+1}-1}{2})^2} \to 0$ as $t \to \infty$;

(ii) $\frac{\sum_{k=q}^{t-1} 3^{t-k-1}(\#V_{k+1}-1)}{t(\#V_t)^2} \le \frac{\sum_{k=q}^{t} 3^{t-k-1}(\#V_k)^2}{t(\#V_t)^2} \le \frac{1}{4} \frac{3^t \sum_{k=0}^{t} 3^{k+1}}{t(\frac{3^{t+1}-1}{2})^2} \to 0$ as $t \to \infty$.

Therefore we obtain that

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\bar{D}(t)}{t} &= \limsup_{t\to\infty} \frac{\pi_t}{t(\#V_t - 1)\#V_t/2} \\
&\le 6\alpha^* \lim_{t\to\infty} \frac{\sum_{k=q}^{t-1} 3^{t-k-1} k(\#V_k)^2}{t(\#V_t - 1)\#V_t/2} \\
&= 12\alpha^* \lim_{t\to\infty} \frac{\sum_{k=q}^{t-1} 3^{t-k-1} k(\#V_k)^2}{t(\#V_t)^2}.
\end{aligned}$$

Using Stolz theorem, we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{\sum_{k=q}^{t-1} 3^{t-k-1} k(\#V_k)^2}{t(\#V_t)^2} &= \frac{1}{3} \lim_{t\to\infty} \frac{\sum_{k=q}^{t-1} k(\#V_k)^2/3^k}{t(V_t)^2/3^t} \\
&= \frac{1}{3} \lim_{t\to\infty} \frac{(t-1)(\#V_{t-1})^2/3^{t-1}}{t(\#V_t)^2/3^t - (t-1)(\#V_{t-1})^2/3^{t-1}} \\
&= \lim_{t\to\infty} \frac{(t-1)(3^t-1)^2}{6(t\cdot 3^{2t}) + 3^{2t+1} - 6\cdot 3^t - 2t + 3} \\
&= \frac{1}{6}.
\end{aligned}$$

That means

$$\limsup_{t\to\infty} \frac{\bar{D}(t)}{t} \le 2\alpha^*. \tag{5.13}$$

(ii) Lower bound of π_t : By (5.11), suppose there exists an integer k_0 such that $\frac{\kappa_{t-1}}{t-1} \ge (\alpha^* - \varepsilon)$ for all $t \ge k_0$. Using (5.5) and (5.10) we have

$$\begin{aligned}
\pi_t &\ge 3\pi_{t-1} + \nu_t \\
&\ge 3\pi_{t-1} + 6(\alpha^* - \varepsilon)(\#V_{t-1})^2(t-1) \\
&\ge 3^2\pi_{t-2} + 6(\alpha^* - \varepsilon)\left((\#V_{t-1})^2(t-1) + 3(\#V_{t-2})^2(t-2)\right) \\
&\ge \cdots \ge 6(\alpha^* - \varepsilon)\sum_{k=k_0}^{t-1} 3^{t-1-k} k(\#V_k)^2.
\end{aligned}$$

In the same way as above, we obtain that

$$\liminf_{t\to\infty} \frac{\bar{D}(t)}{t} \ge 2\alpha^*. \tag{5.14}$$

It follows from (5.13) and (5.14) that

$$\lim_{t\to\infty} \frac{\bar{D}(t)}{t} = 2\alpha^*.$$

□

6. Determination of α^*

6.1. **Normal decomposition.**

Given a word σ, let $C(\sigma)$ be the set of letters appearing in σ, $\#\sigma$ the cardinality of $C(\sigma)$, and $\sigma|_{-1}$ the last letter of σ.

For $\sigma = 222113112312$, $\omega(\sigma) = 4$, we can obtain a decomposition

$$\sigma = (222)(11311)(23)(12) = \tau_1\tau_2\tau_3\tau_4$$

such that τ_2, τ_3, τ_4 contains 2 letters and $3\tau_4$, $1\tau_3$, $2\tau_2$ contains 3 letters, where 3, 1 and 2 are the last letter of τ_3, τ_2 and τ_1 respectively. In the same way, for σ with $\omega(\sigma) = l > 1$, we have the decomposition

$$\sigma = \tau_1\tau_2\cdots\tau_{l-1}\tau_l \text{ with } \omega(\sigma) = l > 1 \tag{6.1}$$

satisfying

$$\begin{aligned} &\#\tau_1 \le 2 \text{ and } \#\tau_i = 2 \text{ for } i \ge 2, \\ &|\tau_1| \ge 1 \text{ and } |\tau_i| \ge 2 \text{ for } i \ge 2, \\ &\{\tau_i|_{-1}\} = \{1,2,3\}\backslash C(\tau_{i+1}), \end{aligned} \tag{6.2}$$

where the last one means the tail of τ_i with $i < l$ is *uniquely determined* by τ_{i+1}. For σ with $\omega(\sigma) = 1$, we have $\#\sigma \le 2$, we give the decomposition

$$\sigma = \tau_1 \text{ with } \omega(\sigma) = 1 \tag{6.3}$$

and (6.2) also holds. We call the decomposition (6.1) or (6.3) the **normal decomposition** if (6.2) holds. For $k \ge 3$, let

$$\begin{aligned} T(k) &= \#\{|\tau| = k \text{ with letters in } \{1,2\} : \#\tau = 2\}, \\ h(k) &= \#\{|\tau| = k \text{ with letters in } \{1,2,3\} : \tau|_{-1} = 1 \text{ and } \#\tau \le 2\}, \\ M(k) &= \#\{|\tau| = k \text{ with letters in } \{1,2\} : \tau|_{-1} = 1 \text{ and } \#\tau = 2\}, \\ e(k) &= \#\{|\tau| = k \text{ with letters in } \{1,2,3\} : \#\tau \le 2\}. \end{aligned}$$

Then

$$\begin{aligned} T(k) &= 2^k - 2, \quad h(k) = 2^k - 1, \\ M(k) &= 2^{k-1} - 1, \; e(k) = 3 \cdot 2^k - 3 = 3h(k). \end{aligned}$$

Notice that $e(k) = \#\{|\tau| = k : \omega(\tau) = 1\}$ and

$$2M(k) = T(k). \tag{6.4}$$

Given $k_1 + \cdots + k_l = t$ with $k_1 \ge 1$ and $k_2, \cdots, k_l \ge 2$, consider

$$W_{k_1\cdots k_l} = \#\{|\sigma| = t : \sigma = \tau_1\tau_2\cdots\tau_{l-1}\tau_l \text{ are normal with } |\tau_i| = k_i \text{ for all } i\}.$$

Lemma 6. *If $l \ge 3$, then*

$$W_{k_1\cdots k_l} = h(k_1)\left[(C_2^1 M(k_2))\cdots((C_2^1 M(k_{l-1}))\right](C_3^2 T(k_l)). \tag{6.5}$$

For $l = 2$, we have $W_{k_1k_2} = h(k_1)(C_3^2 T(k_l))$.

Proof. Fix $l \ge 3$. At first we can choose two distinct letters $i_1 < i_2$ from $\{1,2,3\}$ such that $C(\tau_l) = \{i_1, i_2\}$, then the number of choices for τ_l is $C_3^2 T(k_l)$. When τ_l is given, the tail of τ_{l-1}, say 1, is uniquely determined by τ_l, then the number of choices for τ_{l-1} is $C_2^1 M(k_{l-1})$. Again and again, when τ_2 is given, then the tail of τ_1 is uniquely determined and number of choices for τ_1 is $h(k_1)$. Then (6.5) follows. $\square$

Then we have

$$\sum_{l\geq 1}\sum_{\substack{k_1\geq 1,k_2,\cdots k_l\geq 3\\ k_1+\cdots+k_l=t}} W_{k_1\cdots k_l}=e(t)+\#\{|\sigma|=t:\omega(\sigma)\geq 2\}=3^t,$$

and

$$\sum_{|\sigma|=t}\omega(\sigma)=e(t)+\sum_{l\geq 2}\left(l\cdot\sum_{\substack{k_1\geq 1,\ k_2,\cdots,k_l\geq 2\\ k_1+\cdots+k_l=t}} W_{k_1\cdots k_l}\right).$$

Lemma 7. *If $l\geq 2$, then*

$$W_{k_1\cdots k_{l-1}k_l}=T(k_l)W_{k_1\cdots k_{l-1}}. \tag{6.6}$$

Proof. If $l\geq 3$, using (6.4) and (6.5), we obtain (6.6). If $l=2$, we have

$$W_{k_1k_2}=h(k_1)(C_3^2T(k_2))=T(k_2)(3h(k_1))=T(k_2)(e(k_1))=T(k_2)W_{k_1} \tag{6.7}$$

since $3h(k)=e(k)$. □

6.2. **Proof of Proposition 4.**

For $x=\cdots x_2x_1\in\Sigma$, let $x|_{-k}=x_kx_{k-1}\cdots x_1$.

Given $k_1\geq 1$ and $k_2,\cdots,k_q\geq 2$, consider

$$A_{k_1\cdots k_q}=\{x\in\Sigma:(x1)|_{-(k_1+\cdots+k_q)}=\tau_1\tau_2\cdots\tau_{q-1}\tau_q \text{ is a normal decomposition with } |\tau_i|=k_i \text{ for all } i\}.$$

Since 1 is the tail of τ_q, we have

$$\mu(A_{k_1\cdots k_q})=\frac{W_{k_1\cdots k_q}/3}{3^{k_1+\cdots+k_q-1}}=\frac{W_{k_1\cdots k_q}}{3^{k_1+\cdots+k_q}}. \tag{6.8}$$

Suppose $x1=\cdots x_{p_{n+1}}\cdots x_{p_n}\cdots x_{p_1}\cdots 1$ and $Y_t(x)=n$, then

$$(x1)|_{-t}=(x_t\cdots x_{p_n})(x_{(p_n-1)}\cdots x_{p_{n-1}})\cdots(x_{p_2}\cdots x_{p_1})(x_{(p_1-1)}\cdots 1)$$

with $p_{n+1}>t\geq p_n$ and

$$x\in A_{k_1k_2\cdots k_{n+1}} \text{ with } k_1+\cdots+k_{n+1}=t \text{ and } Y_t(x)=n \tag{6.9}$$

where $k_1=t-p_n+1(\geq 1)$ and $k_i=p_{n-i+2}-p_{n-i+1}(\geq 2)$ for $i\geq 2$.

Lemma 8. *Suppose $J_n=S_1+S_2+\cdots+S_n$ and $Y_t=\sup\{n:J_n\leq t\}$ are defined in Section 2. Then*

$$\liminf_{t\to\infty}\frac{\sum_{|\sigma|=t}d(\sigma,\emptyset)}{t3^t}\quad\geq\quad\liminf_{t\to\infty}\frac{\sum_{k=2}^{t-1}\mathbb{E}(Y_{t-k})\frac{2^k-2}{3^k}}{t},$$

$$\limsup_{t\to\infty}\frac{\sum_{|\sigma|=t}d(\sigma,\emptyset)}{t3^t}\quad\leq\quad\limsup_{t\to\infty}\frac{\sum_{k=2}^{t-1}\mathbb{E}(Y_{t-k})\frac{2^k-2}{3^k}}{t}.$$

Proof. For $Y_{t'}=\sup\{n:J_n\leq t'\}$, using (6.9) we have

$$\mathbb{E}(Y_{t'})=\sum_q\sum_{\substack{k_1+\cdots+k_q=t'\\ k_1\geq 1,k_2,\cdots,k_q\geq 2}}(q-1)\mu(A_{k_1\cdots k_q}).$$

Using (6.6) and (6.8), we obtain that

$$\begin{aligned}
&\frac{\sum_{|\sigma|=t}\omega(\sigma)}{3^t}\\
=\ &\frac{e(t)}{3^t}+\sum_{k=2}^{t-1}\sum_{l\geq 2}\left(l\cdot\frac{T(k)}{3^k}\sum_{\substack{k_1+\cdots+k_{l-1}=t-k\\ k_1\geq 1,k_2,\cdots,k_{l-1}\geq 2}}\mu(A_{k_1\cdots k_{l-1}})\right)
\end{aligned}$$

where $\frac{e(t)}{3^t}\to 0$. We notice that

$$\begin{aligned}
&\sum_{k=2}^{t-1}\sum_{l\geq 2}\sum_{\substack{k_1+\cdots+k_{l-1}=t-k\\ k_1\geq 1,k_2,\cdots,k_{l-1}\geq 2}} l\cdot\frac{T(k)}{3^k}\mu(A_{k_1\cdots k_{l-1}})\\
\leq\ &2\sum_{k\geq 2}\frac{T(k)}{3^k}+\sum_{k=2}^{t-1}\sum_{l\geq 2}\sum_{\substack{k_1+\cdots+k_{l-1}=t-k\\ k_1\geq 1,k_2,\cdots,k_{l-1}\geq 2}}(l-2)\frac{T(k)}{3^k}\mu(A_{k_1\cdots k_{l-1}})\\
\leq\ &2+\sum_{k=2}^{t-1}\mathbb{E}(Y_{t-k})\frac{T(k)}{3^k}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{k=2}^{t-1}\sum_{l\geq 2}\sum_{\substack{k_1+\cdots+k_{l-1}=t-k\\ k_1\geq 1,k_2,\cdots,k_{l-1}\geq 2}} l\cdot\frac{T(k)}{3^k}\mu(A_{k_1\cdots k_{l-1}})\\
\geq\ &\sum_{k=2}^{t-1}\sum_{l\geq 2}\sum_{\substack{k_1+\cdots+k_{l-1}=t-k\\ k_1\geq 1,k_2,\cdots,k_{l-1}\geq 2}}(l-2)\frac{T(k)}{3^k}\mu(A_{k_1\cdots k_{l-1}})\\
\geq\ &\sum_{k=2}^{t-1}\mathbb{E}(Y_{t-k})\frac{T(k)}{3^k}.
\end{aligned}$$

Notice that $\frac{e(t)}{3^t}\to 0$, then the lemma follows. $\square$

Proof of Proposition 4. Notice that $\frac{\mathbb{E}(Y_i)}{i}\to\frac{2}{9}$ as $i\to\infty$ and

$$\frac{\sum_{k=2}^{t-1}\left((t-k)\cdot\frac{2^k-2}{3^k}\right)}{t}\to 1 \text{ as } t\to\infty,$$

using Lemma 8 we obtain that

$$\begin{aligned}
\alpha^*=\lim_{t\to\infty}\frac{\sum_{|\sigma|=t}\omega(\sigma)}{t3^t}\ &=\ \lim_{t\to\infty}\frac{\sum_{k=2}^{t-1}\mathbb{E}(Y_{t-k})\frac{2^k-2}{3^k}}{t}\\
&=\ \lim_{t\to\infty}\frac{\sum_{k=2}^{t-1}\frac{\mathbb{E}(Y_{t-k})}{t-k}(t-k)\frac{2^k-2}{3^k}}{t}\\
&=\ \lim_{i\to\infty}\frac{\mathbb{E}(Y_i)}{i}=\frac{2}{9}.
\end{aligned}$$

$\square$

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